Simple Expressions for Safety Factors in Inventory Control

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SIMPLE EXPRESSIONS FOR SAFETY FACTORS IN INVENTORY CONTROL

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Abstract

The literature on inventory control discusses many methods to establish the level of decision parameters -like reorder levels or safety factors-, necessary to attain a prescribed service level. In general, however, these methods are not easy applicable: they often use time-consuming iterations, requiring specific software. In particular, large-scale application on huge numbers of stock keeping items are a heavy burden on the computer system.

In this paper, we consider a periodic review fully back-ordered order up-to level *(R,S)*-system with stationary gamma distributed demand, and constant lead time. Two service level constraints are treated simultaneously: the stock-out probability and the fill rate. For the case that, in addition, the demand distribution parameters are known, we

(i) calculate exact safety factors, depending on three model parameters,

(ii) present simple expressions that give nearly exact safety factors.

The latter expressions are valid for a wide range of parameter values; since implementation is straightforward, our method is appropriate for routine operational use.

For unknown demand parameters, estimates obtained from past observations can be plugged in. The behaviour of the resulting order up-to levels is studied by simulation and appears to be quite satisfactory. A comparison with the standard - normality based - approach is made; an indication of the robustness of our method is given.

Our most important message, however, is that this two-step procedure turns out to be applicable to a much wider range of inventory problems; to illustrate this remark, preliminary results on a specific *(R,s,S)*-system are mentioned.

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1. Introduction.

A vast supply of inventory models exists in the literature. Under a cost criterion, these models balance the various interests like inventory holding costs, ordering costs and backordering cost. For the determination of the optimal decision parameters, generally one or more equations have to be solved iteratively, while often numerical integration or other approximating techniques are employed for each iteration. Under a service level criterion the modelling assumptions result in a service equation which must be satisfied in order to attain a prespecified service level. In general, these service equations are not easy to solve either: again, iteration and approximation are employed frequently. Implementing such a procedure requires substantial effort for the practitioner, while the required computer time may be huge, especially since in most inventory situations such procedures are performed regularly and for thousands of items.

Many authors acknowledge the fact that solving service equations or minimizing cost functions iteratively with specially designed software may be a problem for routine operational use. In stead, they advocate heuristic approaches: Silver *et al.* (1998) stress this point at several occasions. A clear example of such heuristics is presented by Ehrhardt (1979) - revised by Ehrhardt and Mosier (1984) and extended by Ehrhardt and Wagner (1982). For the *(s,S)*-system with a cost criterion, he derives the well-known power approximations for the decision parameters. His method is appropriate for routine use, while the exact, but iterative methods provided by Veinott and Wagner (1965) are not. Schneider and Ringuest (1990) expanded this power approximation, using a so-called γ-service level which measures the average backlog relative to the average demand. Other examples are provided by Shore (1986) who derives explicit approximate solutions for some common inventory models based on general approximations for the fractiles and the loss integral of a random variable. Platt *et al.* (1997) also stress the importance of closed-form solutions; they present ‘atheoretic heuristics’ for the order quantity/reorder point *(Q,R)*-system with the P_2-service (‘fill rate’) criterion and normally distributed demand.

The approach adhered to in this paper is new in the area of inventory control: nested regression is applied to express a dependent variable (here: the safety factor) as a function of a minimal set of dimensionless regressors, which choice is based on the analysis of the most simple cases. We apply this approach to an order up-to level policy under periodic review, with deterministic lead time *L*, stationary gamma distributed demand, and under a service level constraint. A *fixed review period R* has great practical advantages and is preferred by management quite often; advantages include lower administration costs, easier coordination of ordering related items, improved workload planning by the buyer and the supplier (often yielding constant lead times), while, finally, the ordering policy can be adapted to changes in the demand pattern at a regular base (Silver *et al.*, 1998). *The gamma distribution* should be used as the modelling distribution for demand, rather than the normal distribution. The obvious reason for this is that the gamma distribution is non-negative and skewed to the right; the first property is imperative, the second is commonly encountered in practice. We will show that the gamma distribution is equally simple to treat theoretically and numerically as the normal. (Another interesting competitor is the Weibull distribution, cf. Tadikamalla (1978) and Zheng and Hayya (1999).) The question to what type of *optimality criterion* the inventory system should be subjected was answered as follows. In general, cost criteria are predominant in US literature. However, performance is usually measured by service level. Furthermore, backorder costs are non-accountable costs for which stock-managers are not held responsible directly by the company. The final reason why we prefer service level constraints to cost criteria is that this choice avoids 'the thorny problem of estimating penalty costs' - Platt *et al.* (1997).

The simplest service criterion is the so-called P_1-criterion: the probability that the net stock drops from a positive value to a negative value during a replenishment cycle should not exceed a prescribed level 1-P_1. The P_2-criterion is used more often in practice; it requires that a prescribed fraction P_2 of total demand is satisfied directly from stock. Note that the main difference is that the P_1-criterion is more qualitative: the *amount* that can not be satisfied from stock is neglected.

Both service criteria will be treated simultaneously here - as far as possible. This simultaneous treatment is enabled by the fact that, for a given type of demand distribution with parameter vector θ, the order up-to (replenishment) level S_i is just a function of θ and P_i (*i*=1,2). As the safety factors $c_i = (S_i - \mu)/\sigma$, *i.e.* the standardized replenishment levels, are even simpler to handle, attention will be focussed on the latter decision parameters. This approach was developed in a previous paper (Strijbosch and Moors, 1998) for normally distributed demand.

We first develop a new theoretical approach, consisting of two steps:

(i) For a gamma demand distribution, exact safety factors are determined, depending on the following three parameters, ν: the coefficient of variation of demand during *R*, P_i: the desired service level, and *k* $(=L/R)$: the lead time *L* expressed in time units of length *R*.

(ii) Simple expressions are derived that result into nearly exact safety factors for any combination of parameter values within a wide range.

Bottleneck for the practical application of these results is ν: in practice, the demand parameters will be unknown. Therefore, our two next steps are:

(iii) Estimates - obtained from past experience - are plugged into our formulae.

(iv) The behaviour of the obtained safety factors is studied in this usual situation of unknown demand parameters.

To our knowledge, the exact safety factors in (i) were never calculated before explicitly. The approximations derived in (ii) are so close that deviations from the desired service levels are at most 3‰ - in the situations considered. The simulations in (iv) indicate that the formulae with estimated demand parameter perform quite satisfactorily - unless the true demand distribution deviates significantly from the gamma distribution.

The paper is organized as follows. Section 2 describes the exact determination of safety factors for the considered inventory models. In Section 3 approximating formulae are developed for the safety factors, and the loss of performance (service) is studied when the resulting approximate values are used in stead of the exact ones. In Section 4 we consider the situation that the parameters of the demand distribution are unknown. In a Monte Carlo investigation, demands are generated from a gamma distribution and the demand parameters are estimated by Simple Exponential Smoothing. Section 5 investigates the effect of applying the standard (normality based) approach encountered in many textbooks, while the true demand distribution is gamma. Analogously, Section 6 investigates the robustness of the proposed methods by assuming lognormal demand. In Section 7 we summarize our conclusions and indicate some directions for further research. An important conclusion is that the proposed method could well be applicable to a wider range of inventory control systems; as an illustration, preliminary results are mentioned for the *(R,s,S)*-model discussed by Tijms and Groenevelt (1984).

2. Exact safety factors.

Up to Section 6, it will be assumed that demand has a stationary gamma distribution. More precisely, we assume that demand follows a (stationary) $\Gamma(\lambda,\rho)$-*process*, meaning that

- demand X_t during any interval of length *t* has distribution $\Gamma(\lambda,\rho t)$,
- demands during disjoint time intervals are independent.

Note that mean μ, variance σ^2 and coefficient of variation ν of the distribution $\Gamma(\lambda,\rho)$ satisfy

$$\mu = \rho/\lambda,\ \ \sigma^2 = \rho/\lambda^2,\ \ \nu = 1/\sqrt{\rho}$$

The service criteria P_1 and P_2 concentrate on the behaviour of the inventory system during a replenishment cycle; hence, an interval of length *R* is taken as time unit for the gamma process. Since, besides, $1/\lambda$ is a scale parameter, we will fix the gamma process for the moment by assuming

$$X_R \sim \Gamma(1,\rho)$$

Denote the corresponding p.d.f. and c.d.f. by f_ρ and F_ρ, respectively, so that

$$f_\rho(x) = \frac{1}{\Gamma(\rho)}\, x^{\rho-1} e^{-x},\ \ x > 0$$

We can express the lead time *L* in units of length *R* without loss of generality (let $k = L/R$); then demand during lead time, and during review plus lead time have the distributions:

$$X_L \sim \Gamma\left(1,\rho_L\right) \quad \text{with } \rho_L = k\rho$$

$$X_{R+L} \sim \Gamma\left(1,\rho_{R+L}\right) \quad \text{with } \rho_{R+L} = \mu_{R+L} = (1+k)\rho$$

The popular - but slightly ambiguous - definition of the service criteria states that P_1 is the probability of no stock-out during a replenishment cycle, while P_2 is the average long-run fraction of demand satisfied from stock on hand. To satisfy these criteria, the corresponding replenishment levels S_i $(i=1,2)$ are standardly chosen according to the equations

$$(2.1) \quad P(X_{R+L} \leq S_1) = P_1$$

$$(2.2) \quad E\left[(X_{R+L} - S_2)^+\right] = (1 - P_2)\,\rho$$

where $Y^+ = \max(Y, 0)$. Note that they relate to the demand during periods of length $R+L$.

However, de Kok (1990) argued that these equations do not take correctly into account the event that the net stock is already negative at the beginning of a replenishment cycle. For $L \leq R$, he advocates the more precise equations

$$(2.3) \quad P(X_{R+L} \leq S_1) + P(X_L > S_1) = P_1$$

$$(2.4) \quad E\left[(X_{R+L} - S_2)^+\right] - E\left[(X_L - S_2)^+\right] = (1 - P_2)\,\rho$$

P_1, for example, is now interpreted as the probability that net stock *drops* from positive to negative during a replenishment cycle, i.e. that a fresh stock-out occurs. Note that the variables X_L and X_{R+L} are dependent, since they relate to overlapping intervals; however, only the marginal distributions are of interest according to (2.3) and (2.4). See for full details de Kok (1990); compare also Tijms (1994, p.58) and Silver *et al.* (1998, p.280). In our paper, the latter pair of service equations will be used in stead of (2.1) and (2.2). Of course, for $L = 0$, both pairs are identical, while for high service levels the numerical differences are only marginal. Throughout, only the situation $L \leq R$ will be considered.

Define the function g: $\mathbb{R} \to \mathbb{R}^+$ by

$$(2.5) \quad g_\rho(x) = \int_x^\infty (z - x) f_\rho(z)\, dz$$

Then the order up-to levels S_i follow from

(2.6) $$\begin{cases} F_{\rho_\upsilon}(S_1) - F_{\rho_{\psi+\upsilon}}(S_1) = 1 - P_1 \\ g_{\rho_{\psi+\upsilon}}(S_2) - g_{\rho_\upsilon}(S_2) = \rho(1 - P_2) \end{cases}$$

For given parameters ρ, R and L, (2.6) is easy to solve; note that g_ρ can simply be calculated from (cf. Fortuin (1980))

$$g_\rho(x) = \rho\left[1 - F_{\rho+1}(x)\right] - x\left[1 - F_\rho(x)\right]$$

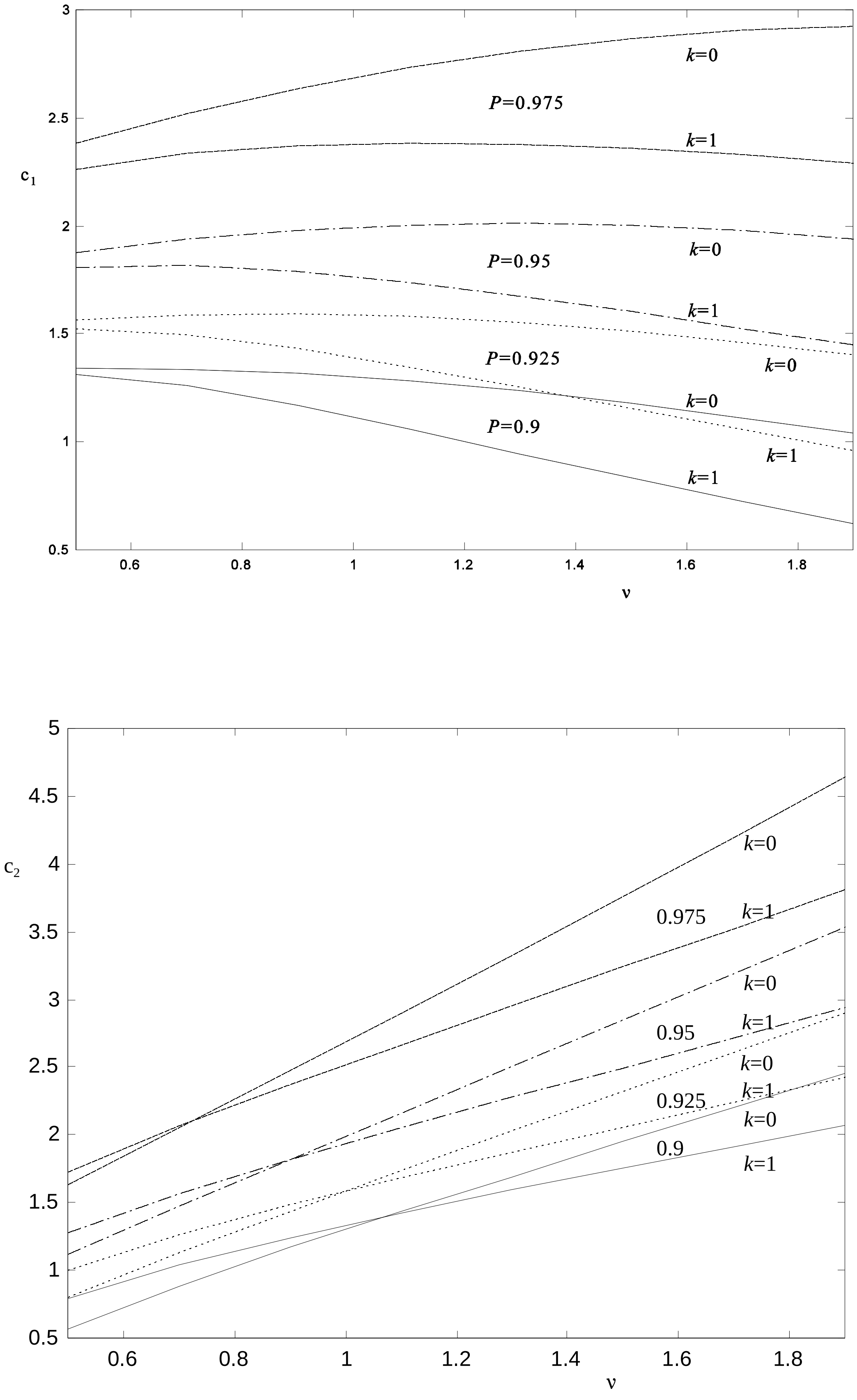
As was already noticed by de Kok (1990), the first equation in (2.6) has two roots, one of them being close to zero. This root corresponds to a very small value of S_1, leading to negative net stock in many (subsequent) replenishment cycles. According to our interpretation of P_1, the larger root is the right one.

The order up-to levels S_i $(i = 1, 2)$ were calculated - using the software package MATLAB - from (2.3) and (2.4) for all combinations of $\nu = 0.5(0.2)1.9$, $P = 0.9(0.025)0.975$ and $k = 0(0.2)1$. These ranges include a wide area of possible application in practice. (Note that for smaller values of ν it is customary to base the order up-to levels on the assumption that demand is normally distributed; see Section 5.) Even more interesting are the safety factors c_i; these standardized replenishment levels now equal

(2.7) $$c_i = \left[S_i - (1+k)\rho\right] / \sqrt{(1+k)\rho}, \quad i = 1,2$$

Appendix A gives all calculated values c_i; Figure 1 shows these values for $k = 0$ $(L = 0)$ and $k = 1$ $(L = R)$. Nomograms like Figure 1 can be used directly in inventory control, if demand follows a stationary gamma process with (approximately) known parameters and if delivery time L $(\le R)$ is known as well.

Figure 1: *Relation between exact safety factors c_i and ν for some values of P and k.*



3. Approximate safety factors.

Calculation of a single safety factor by means of (2.6) is quite fast. Nevertheless, calculation may become time-consuming if a large number of safety factors or replenishment levels have to be calculated. This will often occur in practice, where usually huge amounts of inventory items have to be monitored, and in theoretical applications like in simulation studies. Therefore, we looked for simple approximating formulae for the safety factors c_i.

Our starting point was expression (2.7) for $\lambda = 1$, rewritten as:

$$c_i = \nu S_i / \sqrt{1+k} - \sqrt{1+k}/\nu$$

Now, consider the simplest possible situation: $L = 0$ and $\nu = \rho = 1$; for this exponential distribution, (2.6) leads to the solution

$$S_1 = S_2 = -\ln(1-P_1)$$

So, in the general case with $\rho \neq 0$ and $L > 0$, c_i depends on ν, ν^{-1}, $k^* = \sqrt{1+k}$, $1/k^*$ and $\pi = \ln(1-P)$. To prevent numerical problems in practice, where ν may be close to zero occasionally, ν^{-1} was replaced by $(\nu + 0.05)^{-1}$.

Hence, c_i $(i = 1,2)$ was regressed on the vector $[1 \ \ \nu \ \ (\nu+0.05)^{-1}]$ for all combinations of P and k, the obtained coefficients were regressed on $[1 \ \ \pi]$ for all k, and the subsequently found coefficients were finally regressed on $[1 \ \ k^*]$. (Adding the variable $1/k^*$ appeared to improve the approximation only marginally.) Ample experimentation showed this to be the best choice of regressors. Using all numbers in Appendix A as input, these regressions gave (3.1) as approximating expressions $\bar{c}_i = \bar{c}_i(\nu, P, k)$ for c_i $(i = 1,2)$.

(3.1)

	$0.5 \le \nu \le 1.9$; $0.9 \le P \le 0.975$; $0 \le k \le 1$
$\bar{c}_1$	$\left[1 \;\; \nu \;\; (\nu+0.05)^{-1}\right]\left[A_1\left[1 \;\; k^*\right]^T \; B_1\left[1 \;\; k^*\right]^T\right]\left[1 \;\; \pi\right]^T$
$\bar{c}_2$	$\left[1 \;\; \nu \;\; (\nu+0.05)^{-1}\right]\left[A_2\left[1 \;\; k^*\right]^T \; B_2\left[1 \;\; k^*\right]^T\right]\left[1 \;\; \pi\right]^T$

where

$$A_1 = \begin{bmatrix} 0.508 & -0.246 \\ -1.215 & -0.039 \\ -0.340 & 0.341 \end{bmatrix}; \; B_1 = \begin{bmatrix} -0.614 & -0.096 \\ -0.589 & 0.224 \\ -0.002 & 0.084 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} -0.784 & 0.647 \\ -0.489 & 0.067 \\ -0.618 & 0.174 \end{bmatrix}; \; B_2 = \begin{bmatrix} 0.173 & -0.356 \\ -1.239 & 0.549 \\ -0.285 & 0.158 \end{bmatrix}.$$

By way of example, for $\nu=0.9$, $P=0.95$, $k=0.4$ Appendix A and (3.1) give the values

$$c_1=1.892 \qquad c_2=1.830$$

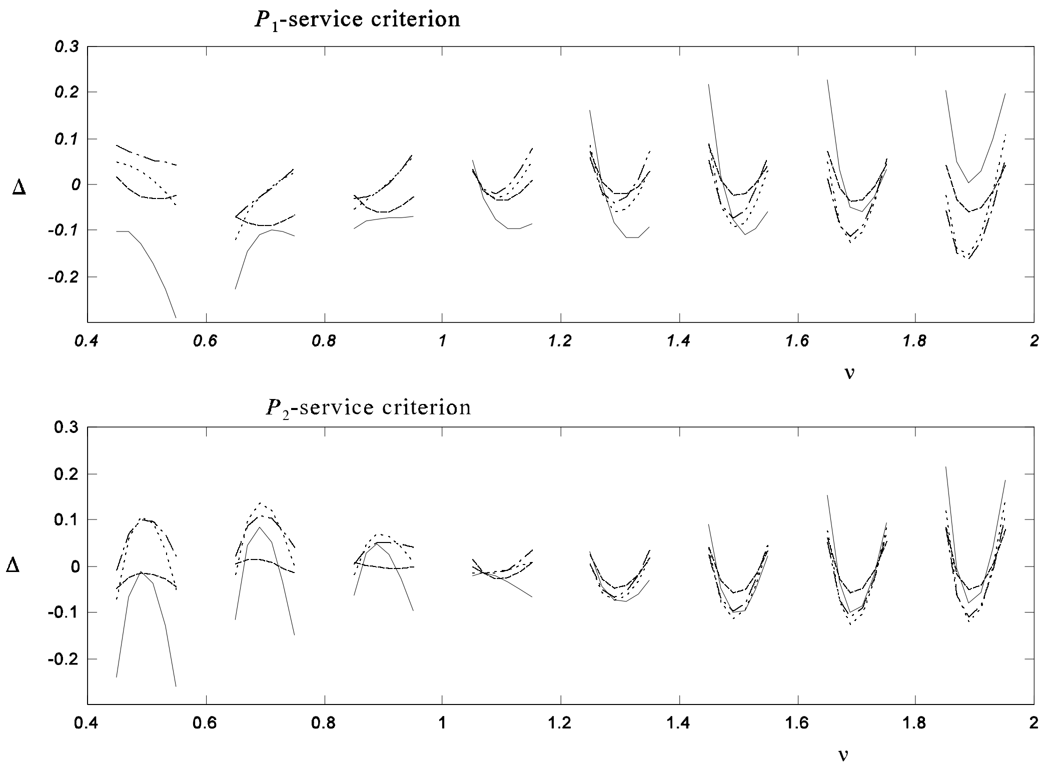
$$\bar{c}_1=1.895 \qquad \bar{c}_2=1.821$$

The value of the determination coefficient between c_1 and $\bar{c}_1$ was 0.9994, and between c_2 and $\bar{c}_2$ even 0.9998. For all figures in Appendix A, the maximum relative deviation $100(c_i-\bar{c}_i)/c_i$ ranges from -2.5 to 3.7% for the P_1-criterion and from -3.5 to 1.3% for P_2.

Of even more importance is the attained service level $\bar{P}_i$, when $\bar{c}_i$ is used in stead of the exact c_i. We will call this the *performance* of a method; it will be ‘measured’ here by the (relative) difference $\Delta_i = 100(P_i - \bar{P}_i)$ where $\bar{P}_i$ is obtained from (2.6) by plugging in

$\bar{S}_i = (1+k)/\nu^2 + \bar{c}_i(\nu, P, k)\sqrt{1+k}/\nu$ - compare (2.7). Of course, optimal performance is reached for $\Delta = 0$. Figure 2 shows the results. For a given value ν, varying values of Δ for k=0(0.2)1 are evenly stretched from ν-0.05 (k=0) to ν+0.05 (k=1); the same line types as in Figure 1 are used again to denote the various P-values.

Figure 2: *Performances (Δ) of $\bar{c}_i$ (3.1) for varying values of ν, P and k.*



The most important conclusion from Figure 2 is that for all tested parameter combinations the difference between the desired service level and the attained service level is within ±0.3%. Note that a *negative* performance means a *higher* attained service level than desired and that the largest differences occur for the lowest service level. So, for given parameters, the approximate service level comes very close to the desired one, while the calculation of the safety factors is extremely fast: in a MATLAB-environment on the average more than 200 times faster than direct calculation of (2.6) using standard functions.

For general gamma distributions $\Gamma(\lambda, \rho)$, within the indicated ranges of P, ν and k, approximate order up-to levels $\bar{S}_i$, $i = 1, 2$ are obtained from:

$$(3.2)\quad \overline{S}_i = \left[(1+k)/\nu^2 + \overline{c}_i(\nu, P, k)\sqrt{1+k}/\nu\right]/\lambda = \mu\,(1+k) + \overline{c}_i(\nu, P, k)\,\sigma\sqrt{1+k}, \quad i = 1, 2$$

4. Unknown demand parameters.

In this section, we will investigate by simulation the performance of the decision rules (3.1) for service equations (2.3) and (2.4) when the demand parameters have to be estimated. Assuming $\mu = 10$ throughout, for each value of $\nu \in \{0.5\,(0.2)\,1.9\}$ and $k \in \{0(0.2)1\}$ two time series $\{x_{L,n}\}$ and $\{x_{R-L,n}\}$ were independently generated from the corresponding gamma distributions

$$(4.1)\quad \begin{cases} X_L \sim \Gamma(\lambda, k\rho) \\ X_{R-L} \sim \Gamma(\lambda, (1-k)\rho) \end{cases}$$

respectively, where *n* runs from -1,000 to 10,000, using the part -1,000 to 0 for the initialisation of the processes. The demand during review time, X_R, and demand during review plus lead time, X_{R+L}, can be found from these samples by addition.

There are several ways for data collection in practice. In theory, we should gather information on the distributions of X_{R+L} and X_L. However, in practice it is customary to check inventory on review moments only. (Note that in our case of a stationary gamma demand process this induces no loss of information.) Therefore, we assume that demand

$$x_n = x_{R,n} = x_{L,n} + x_{R-L,n}$$

during review periods is observed only. From these observations alone, the demand parameters have to be estimated. The usual estimation procedure is simple exponential smoothing (SES), to allow for non-stationarity; we followed this practice to find estimates $M_n = M_{R,n}$ and $V_n = V_{R,n}$ for the mean and variance of demand during review time:

$$(4.2) \quad \begin{cases} M_n = \alpha x_n + (1-\alpha) M_{n-1} \\ V_n = \omega (x_n - M_{n-1})^2 + (1-\omega) V_{n-1} \end{cases}$$

Hence, $D_n = D_{R,n} = \sqrt{V_n}$ is used to estimate the standard deviation, in stead of the more usual MAD; the reason is that the factor 1.25 used to transform the MAD to a standard deviation is not valid for non-normal distributions. Indeed, limited simulations (not presented) show that using the smoothed variance instead of the smoothed MAD leads to substantially better performances.

Next, the corresponding estimates for review plus lead time are found as

$$M_{R+L,n} = M_{R,n}(1+k), \quad D_{R+L,n} = D_{R,n}\sqrt{1+k}$$

Note that ν can be estimated by $D_{R,n}/M_{R,n}$ and λ by $M_{R,n}/D_{R,n}^2$. All combinations of the smoothing parameter values $\alpha \in \{0.01, 0.05, 0.10, 0.15\}$ and $\omega \in \{0.01, 0.03, 0.06, 0.09\}$ were tested in the simulation.

In order to be able to establish the realized service level in the simulation process, at each review instant the order up-to levels are determined according to

$$(4.3) \quad \overline{S}_{i,n+1} = M_{R+L,n} + \overline{c}_i(D_{R,n}/M_{R,n}, P_i, k)\, D_{R+L,n}$$

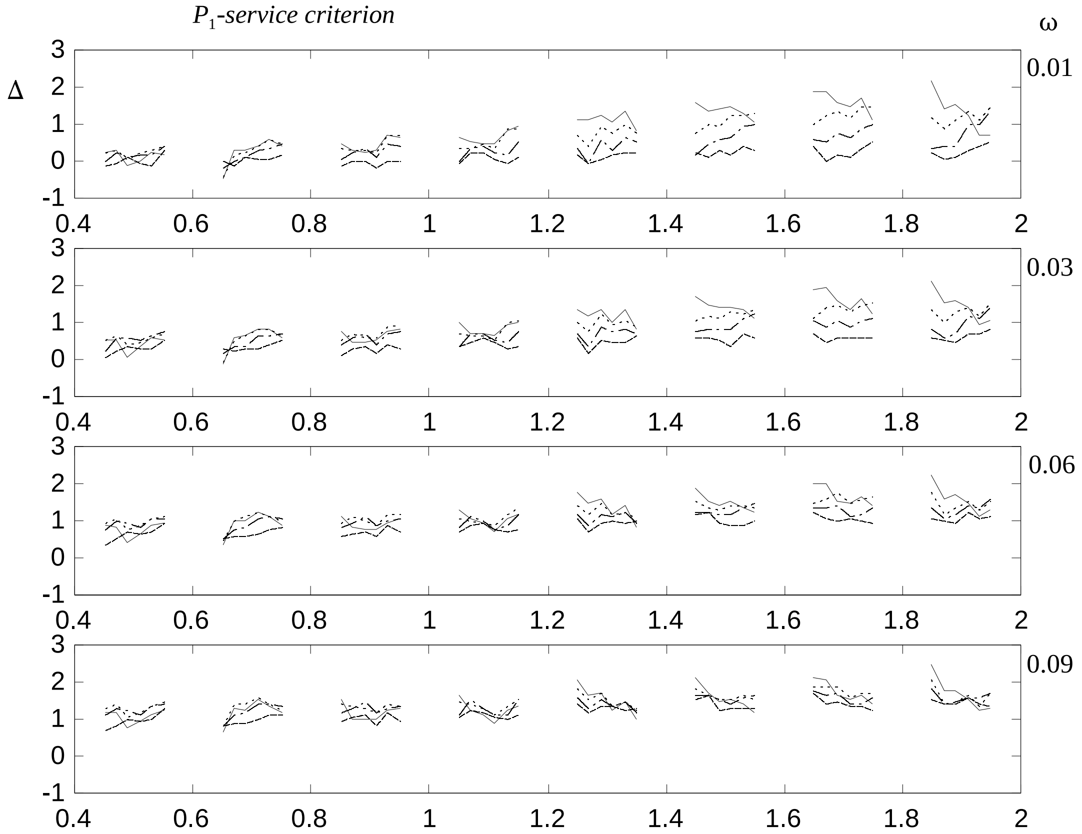
cf. (3.2). Then, at the end of cycle *n* a net stock *NS* of

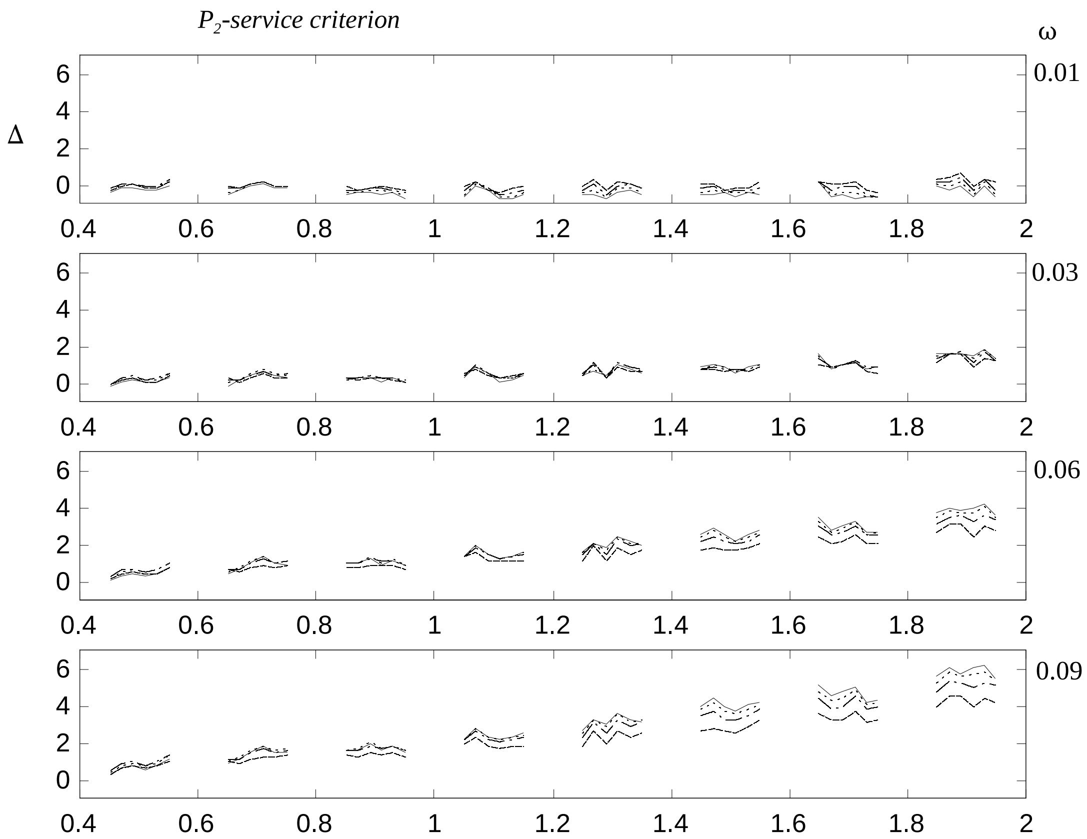
$$NS_{i,n} = \overline{S}_{i,n-1} - \left[x_{L,n-1} + x_{R-L,n-1} + x_{L,n}\right]$$

is obtained. Next, the observed service levels $\hat{P}_i$ over the simulation horizon are obtained according to:

$$(4.4) \quad \hat{P}_1 = 1 - \frac{\#\{(NS_{1,n+1} < 0) \wedge (x_{L,n} \le S_{1,n})\}}{\#\text{cycles}}$$

Figure 3: *Performances (Δ) of $\bar{c}_i$ (3.1) for unknown demand parameters.*





(4.5) $$\hat{P}_2 = 1 - \frac{\left(-NS_{2,n+1}\right)^+ - \left(x_{L,n} - S_{2,n}\right)^+}{\sum x_n}$$

Now for a simulation run of 10,000 cycles (after 1000 cycles to be used as start-up), the differences $\Delta = 100(P_i - \hat{P}_i)$ were calculated. Figure 3 illustrates the resulting performances obtained by the simulation. The results corresponding to the varying values of α are averaged as they differ not much. As in Figure 2, the results for varying values of *k* are stretched around the corresponding value of ν. The main conclusions are:

- The loss of performance due to forecasting is not too serious: up to 2% for the P_1-criterion, up to 6% for P_2.
- A lower performance is obtained in general for higher ν-values, especially for high ω.
- For the P_2-criterion, a clear effect of ω can be seen.
- In general, the loss of performance is larger for lower service levels.

Two remarks regarding the simulation must be made.

- If $\bar{S}_{i,n} < NS_{i,n}$ the complication occurs that the available physical stock is larger than the new order up-to level. Two natural choices emerge: using $\bar{S}_{i,n}$ as the next level indeed, or $NS_{i,n}$. The results in this paper correspond to the first choice. However, the latter choice would not influence the conclusions.
- Incidently, $\bar{S}_{i,n}$ may become much larger than a stock manager would allow. Therefore an upper bound of 100 for $\bar{S}_{i,n}$ is used in the simulations. The effect of this upper bound is practically negligible.

5. Improvement of performance compared to the conventional approach.

Because in many textbooks (cf. Silver *et al.*, 1998) the standard decision rules are based on the assumption that the demand during lead time or review plus lead time is normally distributed, it is of great interest to study the penalty of using those rules when, in fact, demand has a gamma distribution. As stated before, a gamma distribution fits the true demand distribution better than a normal distribution; nevertheless, many managers choose to use the standard approach because of its wide acceptance and ease to use: most textbooks provide tables to determine safety factors for this case.

In this section we study the performance of the standard approach for the specific situation that $L=0$ (i.e. $k=0$) while demand has a gamma distribution. This standard - normality based - approach leads to the safety factors

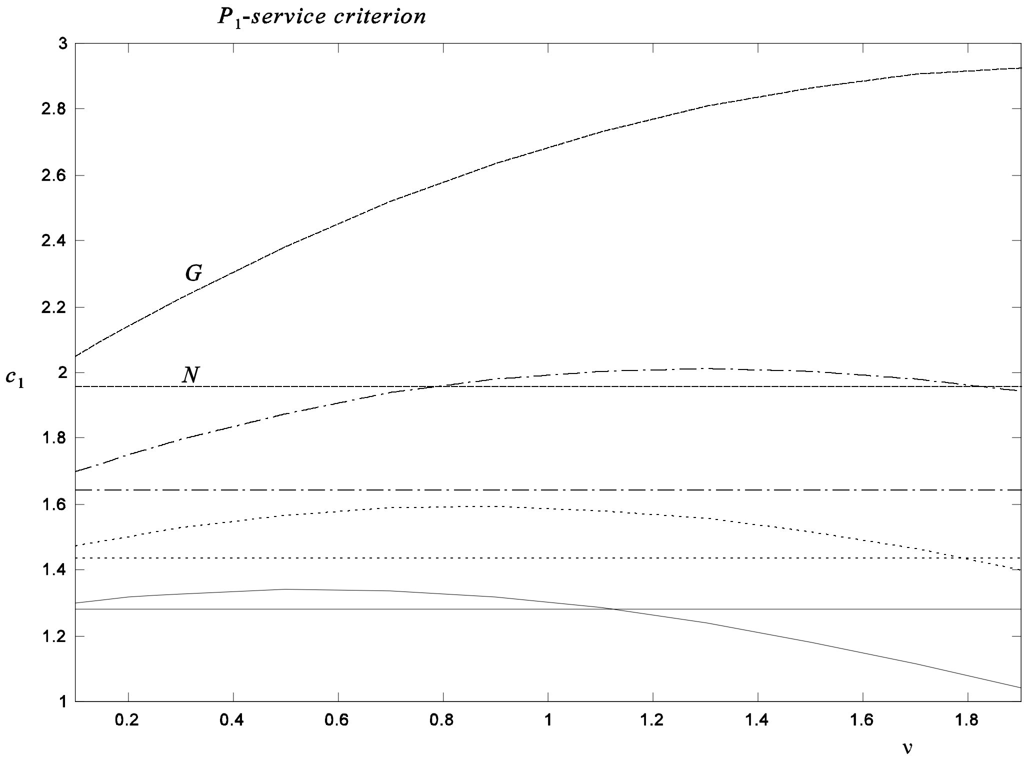
$$(5.1)\quad \begin{cases} c_1 = \Phi^{-1}(P_1) \\ c_2 = G^{-1}\left[(1-P_2)/\nu\right] \end{cases}$$

where $G(k) = \int_k^\infty (z-k)\,\phi(z)\,dz$, and ϕ and Φ are the p.d.f. and c.d.f. of the standard normal distribution. We first show in Figure 4 the exact safety factors for the two demand distributions with the same four *P*-values as in Figure 1. Note that the range of ν-values now starts at 0.1. Since for $\nu \to 0$ the gamma distribution approximates the normal, the safety factors are nearly identical for small ν. However, even for moderate values of ν (e.g. 0.4), safety factors appear to differ substantially. Crucial, however, are the attained service levels. Hence, we repeated our simulation study of Section 4, using now (5.1) in stead of (3.1). Figure 5 shows the performance of this standard approach for $\omega = 0.01$ and 0.09, where, again, the results are averaged over α.

The main conclusions from Figure 5 are:

- The common advice to use the standard approach when the coefficient of variation of demand is below 0.5 seems to be reasonable: the attained service is at most 2% lower than desired. However, for larger values of ν, depending on the value of ω, the attained service level may be 4 (P_1) to 18% (P_2) too low!
- The P_2-performance is nearly identical for various values of the desired service level. The P_1-performance however, is slightly dependent of the desired level.

Figure 4: *Exact safety factors for gamma (G) and normal (N) demand distribution ($L = 0$).*



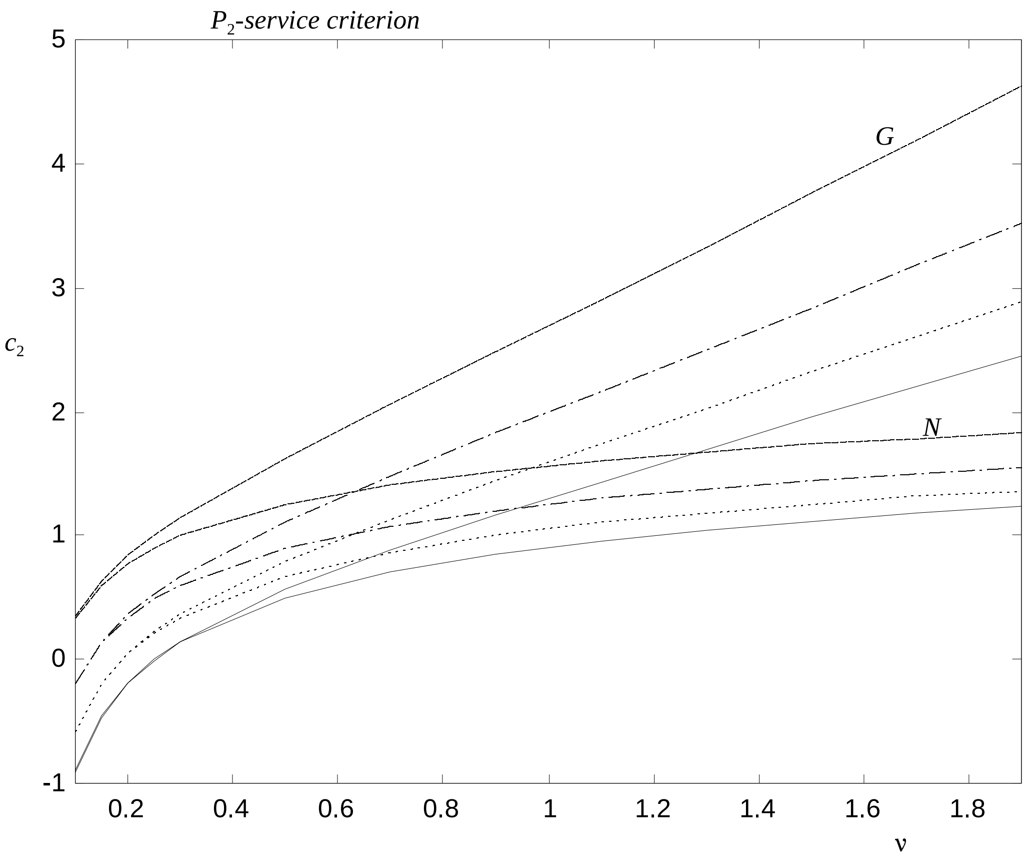
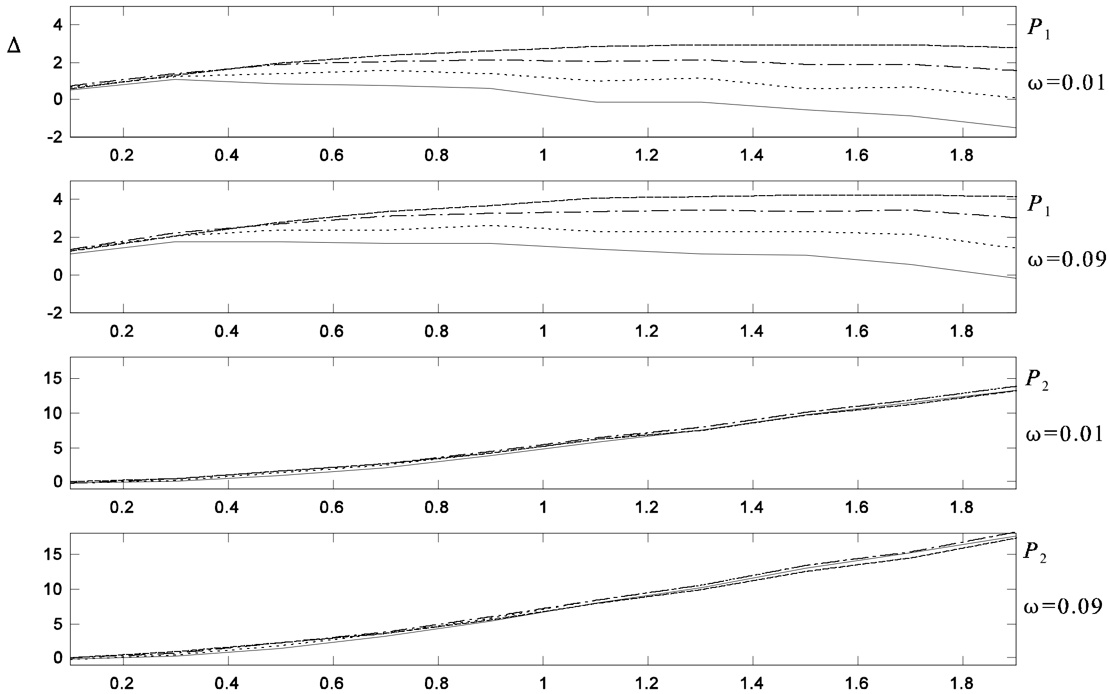


Figure 5: *Performances (Δ) of standard approach; $L=0$; gamma demand.*



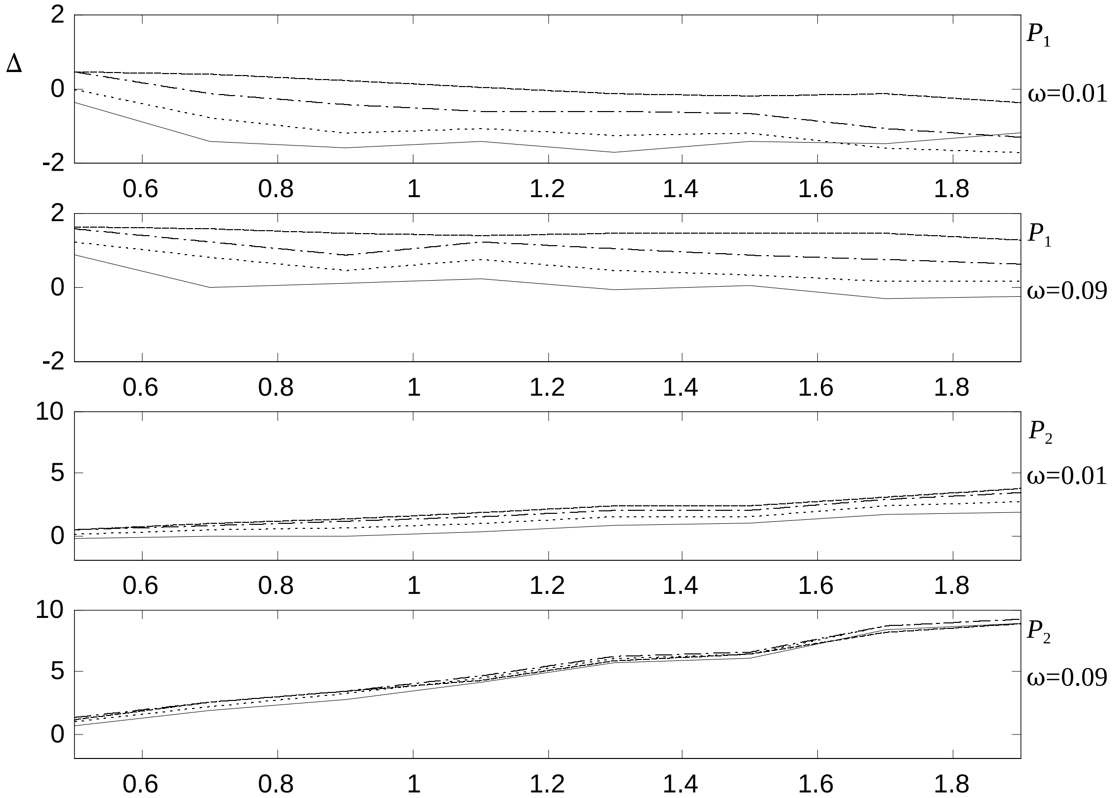
6. Performance when demand has a lognormal distribution.

Another obvious candidate to describe stochastic demand is the lognormal distribution: like the gamma distribution, it is skewed to the right and takes positive values only. Therefore, in this section we investigate the performance of our gamma-based rule (3.1), where in fact demand has a lognormal distribution. Again, L=0 here.

So, the simulation of Section 4 was repeated once again, where now demands X_R are generated from lognormal distributions. Figure 6 shows the results for $\omega=0.01$ and 0.09, averaged over α. The main finding is that attained service levels fall short at most 2% (P_1) to 9% (P_2), the latter value occurring for combined high ν- and ω-values only. This indicates that in many cases, it may be safe to use our proposed methodology even if demand is not really gamma distributed.

Some other demand distributions were considered too. We found that in some cases the attained service levels are even *higher* than desired, e.g. for certain bimodal mixed Erlang distributions.

Figure 6: *Performances (Δ) of rule (3.1), $L=0$; lognormal demand.*



7. Conclusions and discussion.

We considered the familiar single-item (*R,S*)-control policy under the P_1- and P_2-service level constraints. We assumed that lead time *L* is fixed and at most equal to review time *R*, while excess demand is backordered. We argued that there are excellent reasons to model demand by means of the gamma distribution: its theoretical and numerical treatment is equally simple as for the more traditional normal distribution. (To phrase it very radically, the normal distribution should disappear from inventory control literature - except for pedagogical reasons.) For a given coefficient of variation ν of this gamma demand distribution, we calculated exact safety factors c_i for combinations of service level value *P* and lead time to review time ratio $k=L/R$; see Appendix A.

If a given (ν, P, k)-combination does not occur in Appendix A, either direct calculation may be used or a nomogram like Figure 1. For mass application, however, both methods are relatively time-consuming. Hence, we gave in (3.1) simple, but accurate approximations $\bar{c}_i$. Use of (3.1) is at least 200 time faster than direct calculation. The approximations hold for $\nu \in [0.5, 1.9]$, $P \in [0.9, 0.975]$ and $k \in [0, 1]$.

In practice, demand parameters will be unknown and (SES-)estimates are plugged in: our approach was shown to remain satisfactory. As was to be expected, the performance is much better than achieved by the standard, normality based, procedure. Furthermore, gamma based safety factors seem to be rather insensitive to deviations from the gamma distribution assumption.

As to the accuracy of our simulation outcomes, note that the observed service levels $\bar{P}_i$ (Section 3) and $\hat{P}_i$ (Section 4) are based on runs of 10,000 periods of length $R+L$. *If* the observations per replenishment period *were* independent, the standard errors of $\bar{P}_i$ and $\hat{P}_i$ would therefore vary between 0.0016 (P=0.975) and 0.0030 (P=0.9). However, due to the positive dependence between subsequent observations, the actual standard errors will be somewhat higher. Finally, independent demands were generated for all separate ν-values; the consistent behaviour of $\square_i$ and $\hat{P}_i$ as function of ν is a further reassurance of the validity of our main conclusions.

We briefly mention two alternatives to (3.1).

(1) The performance Δ in Figure 2 shows a curvature as function of k. This can partly be eliminated by adding $\square^{*2}$ as regressor, increasing the determination coefficient from 0.9994 to 0.9997 (P_1) and from 0.9998 to 0.99995 (P_2). This improved approximation may be worthwhile for theoretical analyses. However, we do not advocate this alternative for practical applications: note that curvature is already absent in the more practice-oriented Figure 3, while addition of k^{*2} leads to additional columns in the matrices A_i and B_i.

(2) Although (3.1) was obtained from exact c_i for $\nu \in [0.5, 1.9]$ and $P \in [0.9, 0.975]$, the approximation appears to be quite satisfactory for ν-values from 0.3 onwards and for P up to 0.99. But, of course, even better approximations can be found for other parameter intervals similarly. For example, for $\nu \in [0.1, 0.5]$, A_i and B_i in (3.1) should be replaced by

$$A_1 = \begin{bmatrix} 0.025 & 0.179 \\ -1.503 & 0.369 \\ 0.019 & -0.022 \end{bmatrix};\ B_1 = \begin{bmatrix} -0.546 & 0.054 \\ -0.758 & 0.246 \\ 0.007 & -0.006 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} -0.514 & 0.298 \\ -1.217 & 0.783 \\ -0.571 & 0.172 \end{bmatrix};\ B_2 = \begin{bmatrix} -0.242 & -0.076 \\ -0.991 & 0.394 \\ -0.124 & 0.046 \end{bmatrix}.$$

The procedure followed in this paper to establish simple approximate values of safety factors can easily be applied to other inventory models. An interesting candidate is the heuristic (*R, s, S*)- model developed by Tijms and Groenevelt (1984), since it is both frequently referenced in the literature and often used in practice. Indeed, preliminary investigations show that for the P_2-criterion, deterministic lead time and gamma distributed demand, in this situation as well

- exact values of c_2 can be calculated,
- good approximations similar to (3.1) can be found for suitable ranges of P, ν, k and $q = (S - s)/\mu_R$.

The actual service levels attained by this approximation appeared to deviate less than 5‰ from the prescribed ones (in other words: $|\Delta| < 0.5$).

A follow-up paper will present more detailed results and similar findings for other inventory models.

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Appendix A

Below the exact safety factors for the situations that are considered in the paper are tabulated, cf. (2.7). Note that in these tables ν is the coefficient of variation of demand during a period of length *R* (and not *R*+*L*).

Table A.1: Exact safety factors c_1 for varying values of ν, *P* and *k*.

	k	0				0.2			
	P	0.9	0.925	0.95	0.975	0.9	0.925	0.95	0.975
ν	0.5	1.340	1.567	1.877	2.384	1.337	1.559	1.860	2.350
	0.7	1.337	1.589	1.938	2.521	1.324	1.571	1.911	2.473
	0.9	1.318	1.594	1.981	2.638	1.284	1.556	1.934	2.568
	1.1	1.284	1.583	2.006	2.735	1.227	1.521	1.934	2.637
	1.3	1.238	1.556	2.013	2.811	1.159	1.472	1.918	2.686
	1.5	1.180	1.516	2.003	2.867	1.085	1.414	1.888	2.718
	1.7	1.113	1.464	1.980	2.906	1.006	1.348	1.848	2.734
	1.9	1.040	1.402	1.943	2.928	0.925	1.277	1.798	2.738
	k	0.4				0.6			
ν	0.5	1.332	1.550	1.845	2.323	1.326	1.541	1.831	2.300
	0.7	1.308	1.552	1.884	2.432	1.291	1.532	1.860	2.396
	0.9	1.252	1.521	1.892	2.509	1.222	1.489	1.854	2.458
	1.1	1.178	1.469	1.875	2.558	1.135	1.424	1.824	2.492
	1.3	1.094	1.405	1.841	2.587	1.039	1.347	1.778	2.506
	1.5	1.007	1.333	1.797	2.601	0.941	1.265	1.722	2.506
	1.7	0.919	1.257	1.744	2.601	0.845	1.180	1.660	2.494
	1.9	0.831	1.178	1.685	2.592	0.752	1.095	1.593	2.474
	k	0.8				1			
ν	0.5	1.320	1.533	1.819	2.280	1.312	1.524	1.807	2.262
	0.7	1.274	1.514	1.838	2.364	1.258	1.496	1.817	2.336
	0.9	1.194	1.460	1.821	2.413	1.168	1.432	1.790	2.374
	1.1	1.096	1.384	1.780	2.435	1.060	1.348	1.740	2.386
	1.3	0.990	1.298	1.723	2.438	0.946	1.254	1.676	2.379
	1.5	0.884	1.207	1.659	2.426	0.833	1.156	1.604	2.358
	1.7	0.781	1.115	1.589	2.405	0.724	1.058	1.528	2.329
	1.9	0.684	1.025	1.516	2.376	0.622	0.963	1.450	2.293

Table A.2: Exact safety factors c_2 for varying values of ν, P and k.

	k	0				0.2			
	P	0.9	0.925	0.95	0.975	0.9	0.925	0.95	0.975
ν	0.5	0.572	0.803	1.117	1.630	0.637	0.861	1.165	1.660
	0.7	0.889	1.137	1.482	2.058	0.939	1.179	1.512	2.066
	0.9	1.169	1.442	1.826	2.478	1.196	1.460	1.830	2.453
	1.1	1.434	1.737	2.165	2.901	1.433	1.725	2.135	2.836
	1.3	1.693	2.027	2.504	3.328	1.662	1.983	2.438	3.219
	1.5	1.948	2.317	2.844	3.761	1.888	2.239	2.740	3.605
	1.7	2.202	2.607	3.186	4.198	2.112	2.496	3.043	3.994
	1.9	2.456	2.897	3.530	4.638	2.337	2.753	3.348	4.386
	k	0.4				0.6			
ν	0.5	0.687	0.906	1.202	1.683	0.728	0.942	1.232	1.702
	0.7	0.975	1.210	1.533	2.069	1.003	1.233	1.549	2.071
	0.9	1.214	1.472	1.830	2.431	1.227	1.479	1.828	2.411
	1.1	1.431	1.714	2.111	2.784	1.429	1.705	2.089	2.740
	1.3	1.639	1.949	2.386	3.133	1.620	1.921	2.343	3.064
	1.5	1.842	2.180	2.659	3.484	1.806	2.133	2.595	3.387
	1.7	2.044	2.412	2.934	3.837	1.991	2.345	2.847	3.712
	1.9	2.247	2.644	3.210	4.194	2.176	2.558	3.101	4.040
	k	0.8				1			
ν	0.5	0.761	0.972	1.257	1.718	0.789	0.998	1.278	1.731
	0.7	1.025	1.251	1.561	2.071	1.043	1.266	1.571	2.071
	0.9	1.237	1.484	1.825	2.394	1.245	1.488	1.822	2.378
	1.1	1.426	1.696	2.071	2.703	1.423	1.688	2.055	2.670
	1.3	1.604	1.897	2.308	3.006	1.590	1.877	2.278	2.956
	1.5	1.777	2.095	2.542	3.307	1.752	2.062	2.498	3.240
	1.7	1.948	2.291	2.776	3.610	1.912	2.246	2.717	3.524
	1.9	2.119	2.488	3.011	3.915	2.071	2.429	2.937	3.809